

Instability of Ginzburg-Landau Vortices on Manifolds

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Abstract

We investigate two settings of Ginzburg-Landau posed on a manifold where vortices are unstable. The first is an instability result for critical points with vortices of the Ginzburg-Landau energy posed on a simply connected, compact, closed 2-manifold. The second is a vortex annihilation result for the Ginzburg-Landau heat flow posed on certain surfaces of revolution with boundary.

1 Introduction

In this paper we consider the Ginzburg-Landau energy posed on a 2-manifold. We will present two results, one for critical points of the Ginzburg-Landau energy and one for the Ginzburg-Landau heat flow, both showing the non-existence of stable vortex solutions under certain geometric assumptions on the manifold. We say a critical point is unstable if there is a direction in which the second variation of the energy is negative. For the heat flow, we will show that all initial data, even those containing vortices, will eventually converge to a vortex-free solution.

Let E_ε be Ginzburg-Landau energy on an orientable manifold \mathcal{M} equipped with a metric g for $u : \mathcal{M} \rightarrow \mathbb{C}$,

$$E_\varepsilon(u) = \frac{1}{2} \int_{\mathcal{M}} \|\nabla_g u\|_g^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2} dv_g.$$

There is a vast literature on Ginzburg-Landau, but we review here just a few of the results most closely related to our investigation. When \mathcal{M} is a bounded domain $\Omega \subset \mathbb{R}^2$, and under an S^1 -valued Dirichlet condition, Bethuel, Brezis and Hélein establish in [4] that vortices of minimizers converge as $\varepsilon \rightarrow 0$ to a finite set of points or limiting vortices $\{a_i\}$. Here vortices refer to zeros of the order parameter u_ε carrying nonzero degree. Moreover, the limiting vortex locations $\{a_i\}$ will minimize a renormalized energy W . Another important result

comes in [7] where for $u : \mathcal{M} = \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^N$, Jimbo and Morita prove that under homogeneous Neumann boundary conditions, if Ω is convex, the only stable critical points are constants for any $\varepsilon > 0$.

Most important to our work on stability of critical points is the work of Serfaty in [10] on Ginzburg-Landau in simply connected planar domains. Here she shows that there is no nonconstant stable critical point of E_ε with homogeneous Neumann boundary conditions for ε small. To achieve this, she shows that the renormalized energy has no stable critical points. Then using her theory of “ C^2 -Gamma convergence,” she argues that there must be unstable directions for the Hessian of E_ε as well for ε small. Our first main result (Theorem 2.1) in this paper is that for compact, simply connected 2-manifolds without boundary, any critical points must be unstable when ε is small if at least one limiting vortex is located at a point of positive Gauss curvature. Furthermore, if one additionally assumes that \mathcal{M} is a surface of revolution with non-zero Gauss curvature at at least one of the poles, then we argue that all critical points are unstable for ε small, regardless of the curvature of the manifold at the limiting vortex locations (Theorem 2.3). To prove this, we will apply Serfaty’s abstract result in [10] (see Theorem 2.2 below), showing again that the renormalized energy has no stable critical points on such manifolds. For Ginzburg-Landau posed on a 2-manifold, Baraket generalizes the work of [4] to identify the renormalized energy on compact 2-manifolds without boundary in [1]. We should perhaps note that for Ginzburg-Landau in 3-dimensional domains, there do exist stable vortex solutions ([9]). This analysis will be presented in Section 2.

The second setting we consider is the heat flow for the Ginzburg-Landau energy, with $\varepsilon = 1$, on surfaces of revolution \mathcal{M} with boundary:

$$\begin{cases} u_t - \Delta_{\mathcal{M}} u = (1 - |u|^2)u & \text{in } \mathcal{M} \times \mathbb{R}_+, \\ u = e & \text{on } \partial\mathcal{M} \times \mathbb{R}_+, \\ u = u_0 & \text{on } \mathcal{M} \times \{t = 0\}. \end{cases}$$

Here $u : \mathcal{M} \times \mathbb{R}_+ \rightarrow \mathbb{R}^2$, and e is any constant unit vector. We allow the compatible initial data u_0 to have any number of vortices though necessarily the total degree $\sum d_i = 0$ in light of the Dirichlet condition. We want to find conditions on \mathcal{M} such that as $t \rightarrow \infty$, all vortices are annihilated. When $\mathcal{M} = \mathbb{R}^2$, it has been shown in [2] that if u_0 is close to e at infinity in some sense, all vortices of u disappear after a finite time. As in [2], we will derive a Pohozaev-type inequality on surfaces (Lemma 3.3) to prove a similar result when \mathcal{M} is a simply connected surface of revolution satisfying an extra geometric assumption that is unrelated to curvature, see Theorem 3.1. This work is presented in Section 3.

2 Instability of Critical Points on a Compact Surface

In this section we take \mathcal{M} to be a simply connected compact surface without boundary, and g be a metric on \mathcal{M} . Consider the Ginzburg-Landau energy on

\mathcal{M} ,

$$E_\varepsilon(u) = \frac{1}{2} \int_{\mathcal{M}} \|\nabla_g u\|_g^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2} dv_g \quad (2.1)$$

where $u \in H^1(\mathcal{M}, \mathbb{C})$. Let u_ε be the critical point of (2.1), then u_ε satisfies

$$-\Delta_g u_\varepsilon = \frac{u_\varepsilon(1 - |u_\varepsilon|^2)}{\varepsilon^2} \text{ in } \mathcal{M} \quad (2.2)$$

In [4] where \mathcal{M} is a bounded planer domain, Bethuel, Brezis and Hélein prove that under an S^1 -valued Dirichlet boundary condition, critical points u_ε of (2.1) converges to a limiting map u_* strongly in $C_{loc}^k(\mathcal{M} \setminus \bigcup_{i=1}^n a_i)$ for every integer k and in $C_{loc}^{1,\alpha}(\mathcal{M} \setminus \bigcup_{i=1}^n a_i)$ for $\alpha < 1$ where $\{a_i\}$ is a finite set. This result has been generalized to a compact manifold \mathcal{M} without boundary, cf. [1, 6] and has since been refined, see e.g. [8] and [12]. Thus, we have:

Proposition 2.1. *Let $\{u_\varepsilon\}$ be a sequence of critical points of E_ε with $E_\varepsilon(u_\varepsilon) \leq C|\log \varepsilon|$ for some constant $C > 0$. Then up to extraction of a subsequence, there exists a finite set of points a_1, \dots, a_n in \mathcal{M} such that $u_\varepsilon \rightarrow u_*$ strongly in $W^{1,p}(\mathcal{M})$ for $p < 2$ and in $H_{loc}^1(\mathcal{M} \setminus \bigcup_{i=1}^n a_i)$.*

We will refer to these points a_1, \dots, a_n as limiting vortices associates with the sequence $\{u_\varepsilon\}$. The same result holds on a compact manifold with modifications, see Proposition 5.5 in [6].

From the Uniformization Theorem, there is a conformal map $h : \mathcal{M} \rightarrow \mathbb{R}^2 \cup \{\infty\}$, so that the metric g is given by $e^{2f}(dx_1^2 + dx_2^2)$ for some smooth function f . We recall that $\Delta f = -K_{\mathcal{M}} e^{2f}$, where $K_{\mathcal{M}}$ is the Gauss curvature on \mathcal{M} . Then for $U_\varepsilon := u_\varepsilon \circ h^{-1}$, (2.2) transforms to

$$-\Delta U_\varepsilon = \frac{e^{2f}}{\varepsilon^2} U_\varepsilon (1 - |U_\varepsilon|^2) \text{ in } \mathbb{R}^2. \quad (2.3)$$

We may assume that $h(a_i) \neq \infty$ for all i and denote $b_i := h(a_i)$. With a slight abuse of terminology, we will also call the b_i 's limiting vortices. Then u_* is the harmonic map associated to (b_i, d_i) :

$$u_*(x) = \prod_{i=1}^n \left(\frac{x - b_i}{|x - b_i|} \right)^{d_i} e^{i\psi} \text{ in } \mathbb{R}^2, \quad (2.4)$$

where $d_i \in \mathbb{Z} \setminus \{0\}$, $\sum_{i=1}^n d_i = 0$, and ψ is a smooth harmonic function. We also note that the notion of convergence linking $\{u_\varepsilon\}$ to $\{a_i\}$ is that of convergence of the sequence of Jacobians, namely

$$\text{curl}(iU_\varepsilon, \nabla U_\varepsilon) \rightharpoonup 2\pi \sum_{i=1}^n d_i \delta_{b_i}$$

in the sense of distributions, where (\cdot, \cdot) denotes the scalar product in \mathbb{C} . For Euclidean domains, the proof of this convergence of Jacobians can be found in

[8, 12] and the adaptation to the manifold setting is immediate. Moreover, the renormalized energy can be defined in the following way:

Given $\{a_i\}_{i=1}^n \subset \mathcal{M}$, let $B_i^g(r)$ be the geodesic ball in \mathcal{M} centered at a_i with radius r , and $B_i(r) = h(B_i^g(r)) \subset \mathbb{R}^2$. Consider Φ_r which satisfies

$$\begin{cases} \Delta \Phi_r = 0 & \text{in } \mathbb{R}^2 \setminus \bigcup_{i=1}^n B_i(r) \\ \Phi_r = \text{const.} & \text{on each } \partial B_i(r) \\ \int_{\partial B_i(r)} \frac{\partial \Phi_r}{\partial \nu} = 2\pi d_i & \text{for } 1 \leq i \leq n. \end{cases} \quad (2.5)$$

To see the existence of such a solution Φ_r , we first consider a functional

$$E(v) = \int_{\mathcal{M}} |\nabla_g v|^2 - 2\pi \sum_{i=1}^n d_i v|_{\partial B_i^g(r)}$$

defined for $v \in H^1(\mathcal{M})$ such that $v|_{\partial B_i^g(r)} = \text{const.}$ for each i . Using the direct method one can show that there exists a minimizer v_* of E . Then since g is a conformal metric, $v_* \circ h^{-1}$ satisfies (2.5). The renormalized energy is defined by

$$W(\mathbf{b}) = \lim_{r \rightarrow 0} \frac{1}{2} \int_{\mathbb{R}^2 \setminus \bigcup_{i=1}^n B_i(r)} |\nabla \Phi_r|^2 dx - \pi \sum_{i=1}^n d_i^2 \log \frac{1}{r}, \quad (2.6)$$

where $\mathbf{b} = (b_1, \dots, b_n)$.

Finally, using the fact that $\Phi_r(x) \approx \sum_{i=1}^n d_i \log |x - b_i|$ for $x \in \mathbb{R}^2 \setminus \bigcup_{i=1}^n B_i(r)$ when $r \ll 1$, Theorem 2.1 in [1] establishes that W can be written as

$$W(\mathbf{b}) = \pi \sum_{i=1}^n d_i^2 f(b_i) - \pi \sum_{i \neq j} d_i d_j \log |b_i - b_j|. \quad (2.7)$$

Our first main result is the following:

Theorem 2.1. *Let $\{u_\varepsilon\}$ be a family of solutions to (2.3) such that $E_\varepsilon(u_\varepsilon) \leq C|\log \varepsilon|$, and let $\{a_i\}_{i=1}^n \subset \mathcal{M}$ be the limiting vortices for $\{u_\varepsilon\}$. Suppose there exists an a_i such that the Gauss curvature $K_{\mathcal{M}}$ is positive at a_i . Then for ε small enough, u_ε is unstable.*

Remark 2.1. Theorem 2.1 implies that if \mathcal{M} has positive curvature everywhere, then there is no stable solution to (2.3) having vortices for ε small enough. Moreover, in this case any solution without any vortices must then be a constant (see Lemma 3.2 in the next section). For the special case where $\mathcal{M} = \mathbb{S}^\varepsilon$, this instability result was first obtained by Contreras, [5]

Remark 2.2 (The Apple Problem). If one wants to look for an example of a stable nonconstant critical point, one might consider a surface of revolution \mathcal{A} obtained by rotating a smooth curve Γ about the z -axis shown in Figure 1, so that the shape of \mathcal{A} is like an apple. Indeed, one can easily construct a critical point with vortices at S and N (cf. [6]), but it cannot be stable in view of Theorem 2.1, since $K_{\mathcal{A}}$ is positive at poles N and S . We note that for the 3-D case (solid apple in \mathbb{R}^3), it has been proven in [9] that a critical point with a vortex line through S and N is a local minimizer for ε small enough.

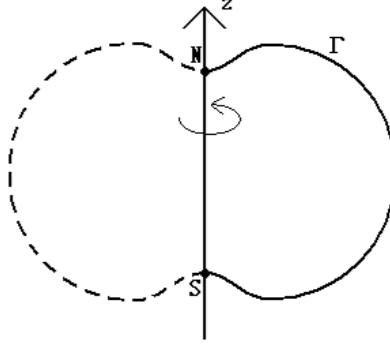


Figure 1: Surface of revolution \mathcal{A}

To prove Theorem 2.1, our main tool will be Serfaty's abstract result in [10] for any C^2 functionals F_ε (resp. F) defined over \mathcal{S} (resp. \mathcal{S}'), which is an open set of an affine space associated to a Banach space \mathcal{B} (resp. \mathcal{B}') satisfying a kind of " C^2 Γ -convergence". Let $u_\varepsilon \in \mathcal{S}$ be a family of critical points of F_ε . Assume u_ε converges to $u \in \mathcal{S}'$ in some topology. Then denoting by n_ε^- (resp. n^-) the dimension of the space spanned by eigenvectors of $D^2F_\varepsilon(u_\varepsilon)$ defined over \mathcal{B} (resp. $D^2F(u)$ defined over \mathcal{B}') associated to negative eigenvalues, the theorem states

Theorem 2.2 ([10]). *Suppose that for any $V \in \mathcal{B}'$, there exists $v_\varepsilon(t) \in \mathcal{S}$ defined in a neighborhood of $t = 0$ such that*

$$v_\varepsilon(0) = u_\varepsilon, \quad (2.8)$$

$$\partial_t v_\varepsilon(0) \text{ is a one-to-one linear map on } \mathcal{B}', \quad (2.9)$$

$$\lim_{\varepsilon \rightarrow 0} \frac{d}{dt} \Big|_{t=0} F_\varepsilon(v_\varepsilon(t)) = \frac{d}{dt} \Big|_{t=0} F(u + tV), \quad (2.10)$$

$$\lim_{\varepsilon \rightarrow 0} \frac{d^2}{dt^2} \Big|_{t=0} F_\varepsilon(v_\varepsilon(t)) = \frac{d^2}{dt^2} \Big|_{t=0} F(u + tV). \quad (2.11)$$

Then for ε small enough, we have $n_\varepsilon^- \geq n^-$.

In the same paper, she applies this result to Ginzburg-Landau in bounded domains in \mathbb{R}^2 with homogeneous Neumann boundary conditions. In a similar manner, to prove Theorem 2.1, we apply this approach to Ginzburg-Landau on surfaces. That is, using the same notation as above we shall prove

Proposition 2.2. *Let u_ε be a family of critical points of E_ε such that $E_\varepsilon(u_\varepsilon) \leq C|\log \varepsilon|$, and b_1, b_2, \dots, b_n be limiting vortices with total degree zero. Then hypotheses (2.8) to (2.11) in Theorem 2.2 hold for $F_\varepsilon = E_\varepsilon$, $F = W$ and $\mathcal{B}' = \mathbb{V} = \{(V_1, V_2, \dots, V_n) : V_i \in \mathbb{R}^2 \forall 1 \leq i \leq n\}$.*

Proof. Let u_ε be a family of critical points of E_ε such that $E_\varepsilon(u_\varepsilon) \leq C|\log \varepsilon|$. Then from results in [1] (see also [6]), there exists $\rho > 0$ small enough such that $B_i^g(\rho)$ are disjoint with $|u_\varepsilon| \geq \frac{1}{2}$ in $\mathcal{M} \setminus \bigcup_{i=1}^n B_i^g(\rho)$ for ε small enough.

The construction of $v_\varepsilon(t)$ is based on Proposition III.1 in [10]. Let $B_i = B_i(\rho) = h(B_i^g(\rho))$. For a given $\mathbf{V} \in \mathbb{V}$, we can define a C^1 family of diffeomorphisms of \mathbb{R}^2 , $\chi_t(x) = x + t\mathbf{X}(x)$, in a neighborhood of $t = 0$ such that \mathbf{X} has compact support in a set $K \subset \subset \mathbb{R}^2$ and

$$\mathbf{X}(x) = V_i \text{ in each } B_i.$$

Then we define $\Phi_{0,t}$ by

$$\Delta \Phi_{0,t} = 2\pi \sum_{i=1}^n d_i \delta_{b_i(t)} \text{ in } \mathbb{R}^2, \quad (2.12)$$

so that $\Phi_{0,t}(x) = \sum_{i=1}^n d_i \log |x - b_i(t)|$, and let $\Phi_{0,0} = \Phi_0$, where $b_i(t) = \chi_t(b_i)$. Then we denote by θ_t^i the polar coordinate centered at $b_i(t)$, and let

$$\psi_t = \sum_{i=1}^n d_i \theta_t^i \circ \chi_t - \sum_{i=1}^n d_i \theta_0^i.$$

Then we have

$$\nabla^\perp \Phi_0 + \nabla \psi_t = \nabla \left(\sum_{i=1}^n d_i \theta_t^i \circ \chi_t \right). \quad (2.13)$$

Finally we define $v_\varepsilon(\chi_t(x), t) = u_\varepsilon(x) e^{i\psi_t(x)}$. With the same argument as in [10], one can show that (2.8) and (2.9) hold for v_ε . Since \mathbf{X} is compactly supported, the domain of integration reduces from \mathbb{R}^2 to a compact set. Consequently, the result of product-estimate derived in [11] used in the original proof can be also applied in our case. Therefore we proceed to verify (2.10).

By the change of variables $y = \chi_t(x)$, we have

$$\begin{aligned} E_\varepsilon(v_\varepsilon) &= \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v_\varepsilon(y)|^2 + \frac{e^{2f(y)}}{2\varepsilon^2} (1 - |v_\varepsilon|^2)^2 dy \\ &= \frac{1}{2} \int_{\mathbb{R}^2} |\nabla(u_\varepsilon e^{i\psi_t})(D\chi_t)^{-1}|^2 + \frac{e^{2f(\chi_t)}}{2\varepsilon^2} (1 - |u_\varepsilon|^2)^2 |Jac \chi_t| dx. \end{aligned} \quad (2.14)$$

Noting that χ_t is the identity map in $\mathbb{R}^2 \setminus K$ and a translation along a constant vector V_i in each B_i , we deduce that in $\mathbb{R}^2 \setminus K$ and $\bigcup_{i=1}^n B_i$,

$$\frac{d}{dt} (D\chi_t)^{-1} = \frac{d^2}{dt^2} (D\chi_t)^{-1} = \frac{d}{dt} |Jac \chi_t| = \frac{d^2}{dt^2} |Jac \chi_t| = 0. \quad (2.15)$$

-Derivative of the potential term:

From (2.15), we derive

$$\begin{aligned}
& \frac{d}{dt}|_{t=0} \int_{\mathbb{R}^2} \frac{e^{2f(\chi_t)}}{4\varepsilon^2} (1 - |u_\varepsilon^2|)^2 |Jac \chi_t| dx \\
&= \int_{K \setminus \bigcup_{i=1}^n B_i} \frac{e^{2f(\chi_t)}}{2\varepsilon^2} (1 - |u_\varepsilon^2|)^2 \nabla f \cdot \mathbf{X} dx \\
&+ \int_{K \setminus \bigcup_{i=1}^n B_i} \frac{e^{2f(\chi_t)}}{2\varepsilon^2} (1 - |u_\varepsilon^2|)^2 \frac{d}{dt}|_{t=0} |Jac \chi_t| dx \quad (2.16)
\end{aligned}$$

Since $\nabla f \cdot \mathbf{X}$ and $\frac{d}{dt}|_{t=0} |Jac \chi_t|$ are bounded in $K \setminus \bigcup_{i=1}^n B_i$, one can apply Lemma 3.2 in [1] which asserts that $\frac{e^{2f}}{\varepsilon} (1 - |u_\varepsilon|^2)^2$ converges to a measure supported on $\bigcup_{i=1}^n b_i$. Hence, we have

$$\frac{d}{dt}|_{t=0} \int_{\mathbb{R}^2} \frac{e^{2f(\chi_t)}}{4\varepsilon^2} (1 - |u_\varepsilon^2|)^2 |Jac \chi_t| dx \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \quad (2.17)$$

Similarly,

$$\frac{d^2}{dt^2}|_{t=0} \int_{\mathbb{R}^2} \frac{e^{2f(\chi_t)}}{4\varepsilon^2} (1 - |u_\varepsilon^2|)^2 |Jac \chi_t| dx \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \quad (2.18)$$

-Derivative of the gradient term:

Using (2.15) we have

$$\begin{aligned}
& \frac{d}{dt}|_{t=0} \frac{1}{2} \int_{\mathbb{R}^2} |\nabla(u_\varepsilon e^{i\psi_t})(D\chi_t)^{-1}|^2 |Jac \chi_t| dx \\
&= \int_K iu_\varepsilon \frac{d}{dt}|_{t=0} \nabla \psi_t \cdot \nabla u_\varepsilon + \nabla u_\varepsilon \frac{d}{dt}|_{t=0} (D\chi_t)^{-1} \cdot \nabla u_\varepsilon dx \\
&+ \frac{1}{2} \int_K |\nabla u_\varepsilon|^2 \frac{d}{dt}|_{t=0} |Jac \chi_t| dx. \quad (2.19)
\end{aligned}$$

Theorem 2 in [1] asserts that u_ε converges to u_* in $H_{loc}^1(\mathbb{R}^2 \setminus \bigcup_{i=1}^n b_i)$. Moreover, $\nabla u_* = \nabla^\perp \Phi_0$. Thus we obtain

$$\begin{aligned}
\frac{d}{dt}|_{t=0} E_\varepsilon(v_\varepsilon) &= \int_K \nabla^\perp \Phi_0 \frac{d}{dt}|_{t=0} (D\chi_t)^{-1} \cdot \nabla^\perp \Phi_0 + \frac{d}{dt}|_{t=0} \nabla \psi_t \cdot \nabla u_\varepsilon dx \\
&+ \int_K |\nabla^\perp \Phi_0|^2 \frac{d}{dt}|_{t=0} |Jac \chi_t| dx + o_\varepsilon(1). \quad (2.20)
\end{aligned}$$

Using (2.15) again and the change of variables $x = \chi_t^{-1}(y)$, for any $0 < r < \rho$, we have

$$\begin{aligned}
\frac{d}{dt}|_{t=0} E_\varepsilon(v_\varepsilon) &= \frac{d}{dt}|_{t=0} \frac{1}{2} \int_{\mathbb{R}^2 \setminus \bigcup_{i=1}^n B_i(r)} |(\nabla^\perp \Phi_0 + \nabla \psi_t)(D\chi_t)^{-1}|^2 |Jac \chi_t| dx + o_\varepsilon(1) \\
&= \frac{d}{dt}|_{t=0} \frac{1}{2} \int_{\mathbb{R}^2 \setminus \bigcup_{i=1}^n B_i(t,r)} |\nabla^\perp \Phi_{0,t}|^2 dy + o_\varepsilon(1). \quad (2.21)
\end{aligned}$$

where $B_i(t, r) = \chi_t(B_i(r))$. The last equality comes from (2.13). Next, define $\Phi_{r,t}$ by

$$\begin{cases} \Delta \Phi_{r,t} = 0 & \text{in } \mathbb{R}^2 \setminus \bigcup_{i=1}^n B_i(t, r) \\ \Phi_{r,t} = \text{const.} & \text{on each } \partial B_i(t, r) \\ \int_{\partial B_i(t, r)} \frac{\partial \Phi_{r,t}}{\partial \nu} = 2\pi d_i & \text{for } 1 \leq i \leq n. \end{cases} \quad (2.22)$$

From Lemma 2.2 in [1] and elliptic estimates, we have

$$\int_{\mathbb{R}^2 \setminus \bigcup_{i=1}^n B_i(t, r)} |\nabla \Phi_{0,t}|^2 dx = \int_{\mathbb{R}^2 \setminus \bigcup_{i=1}^n B_i(t, r)} |\nabla \Phi_{r,t}|^2 dx + o_r(1). \quad (2.23)$$

Then by the definition of W , we obtain

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} E_\varepsilon(v_\varepsilon) &= \frac{d}{dt} \Big|_{t=0} \lim_{r \rightarrow 0} \frac{1}{2} \int_{\mathbb{R}^2 \setminus \bigcup_{i=1}^n B_i(t, r)} |\nabla \Phi_{r,t}|^2 dx + o_\varepsilon(1) \\ &= \frac{d}{dt} \Big|_{t=0} W(\mathbf{b}(t)) + o_\varepsilon(1), \end{aligned} \quad (2.24)$$

hence the desired result (2.10).

The verification of (2.11) is again analogous to the argument found in [10] with appropriate adjustments as were just done in verifying (2.10). \square

Proof of Theorem 2.1. Suppose, by contradiction, that there exists a sequence of stable critical points $\{u_\varepsilon\}$ such that $E_\varepsilon(u_\varepsilon) \leq C|\log \varepsilon|$, and, up to extraction, n limiting vortices b_1, b_2, \dots, b_n with say, $K(b_1) > 0$.

Let $V = (V^1, V^2)$ be an arbitrary vector in \mathbb{R}^2 . Then since we are assuming $n_\varepsilon^- = 0$, in view of Proposition 1 and Theorem 2.2, we must have $n^- = 0$, i.e.

$$\frac{d^2}{dt^2} \Big|_{t=0} W(b_1 + tV, b_2, \dots, b_n) = \sum_{i,j=1,2} \frac{\partial^2 W_1}{\partial x_i \partial x_j}(b_1) V^i V^j \geq 0, \quad (2.25)$$

where

$$W_1(x) = \pi d_1^2 f(x) - \pi \sum_{j=1}^n d_1 d_j \log |x - b_j|.$$

Since the second term of W_1 is harmonic, we have

$$\Delta W_1(b_1) = \pi d_1^2 \Delta f(b_1).$$

Noting that the Gauss curvature at b_1 is given by

$$0 < K(b_1) = -\frac{\Delta f}{e^{2f}}(b_1),$$

we deduce that $D^2 f(b_1)$ has at least one negative eigenvalue, which contradicts (2.25). Hence, if u_ε are stable, the number of limiting vortices is 0, i.e. for ε small enough, $|u_\varepsilon| \geq \frac{1}{2}$ in \mathcal{M} . However, as was mentioned in Remark 2.1, this implies that u_ε is a constant. \square

Now, let \mathcal{M} be the surface obtained by rotating a regular curve

$$\gamma(s) = (\alpha(s), 0, \beta(s)), \quad 0 \leq s \leq l, \quad \alpha(s) > 0 \text{ for } s \neq 0, l.$$

about the z -axis, where s is the arc length, i.e. $|\gamma'| = 1$. Furthermore, make the assumptions:

$$\alpha(0) = \alpha(l) = \beta'(0) = \beta'(l) = 0, \text{ and either } \beta''(0) \neq 0 \text{ or } \beta''(l) \neq 0. \quad (2.26)$$

We will henceforth assume $\beta''(0) \neq 0$, the other case being similar. Denoting by θ the rotation angle, we then have a parametrization of \mathcal{M}

$$\mathbf{x}(s, \theta) = (\alpha(s) \cos(\theta), \alpha(s) \sin(\theta), \beta(s)), \quad 0 \leq s \leq l, \quad 0 \leq \theta \leq 2\pi, \quad (2.27)$$

and the induced metric

$$g_{\mathcal{M}} = ds^2 + \alpha^2(s) d\theta^2.$$

Note in particular that, for $\mathcal{M} = \mathcal{S}^2$, we have $\alpha(\phi) = \sin(\phi)$ and

$$g_{\mathcal{S}^2} = d\phi^2 + \sin^2(\phi) d\theta^2.$$

Consider a map $F : \mathcal{S}^2 \rightarrow \mathcal{M}$ such that parameter values (ϕ, θ) corresponding to a point $p \in \mathcal{S}^2$ are mapped to parameter values $(S(\phi), \theta)$ in (2.27) corresponding to the point $q = \mathbf{x}(S(\phi), \theta)$, where $S(0) = 0$ and $S(\pi) = l$. Then F is conformal if S solves

$$S'(\phi) \sin(\phi) = \alpha(S(\phi)) \quad (2.28)$$

Finally, we reparametrize \mathcal{M} by defining $\mathbf{y} : \mathbb{R}^2 \cup \{\infty\} \rightarrow \mathcal{M}$ through

$$\mathbf{y}(x_1, x_2) = (\alpha(S(\phi)) \cos(\theta), \alpha(S(\phi)) \sin(\theta), \beta(S(\phi))) \text{ for } (x_1, x_2) \in \mathbb{R}^2 \cup \{\infty\},$$

where

$$\begin{aligned} \phi &= \cos^{-1}\left(\frac{r^2 - 1}{1 + r^2}\right) \in [0, \pi], \\ \theta &= \tan^{-1}\left(\frac{x_2}{x_1}\right), \\ r^2 &= x_1^2 + x_2^2. \end{aligned} \quad (2.29)$$

In other words, $\mathbf{y} = F \circ P^{-1}$, where P is the stereographic projection from \mathcal{S}^2 to the $x_1 x_2$ plane. Using (2.28) and (2.29) the induced metric is given by

$$\tilde{g}_{\mathcal{M}} = \alpha^2(S(\phi)) \frac{1}{r^2} (dx_1^2 + dx_2^2) \equiv e^{2f} (dx_1^2 + dx_2^2),$$

i.e.

$$f = \ln(\alpha(S(\phi)) \frac{1}{r}). \quad (2.30)$$

Let $A = (\alpha'(S(\phi)) + 1)\frac{1}{r^2} \geq 0$, $B = -\frac{\alpha^2(S(\phi))}{r^2}K_{\mathcal{M}}$. With a direct calculation we obtain

$$D^2f = \begin{pmatrix} -A + \frac{x_1^2}{r^2}(2A + B) & \frac{x_1x_2}{r^2}(2A + B) \\ \frac{x_1x_2}{r^2}(2A + B) & -A + \frac{x_2^2}{r^2}(2A + B) \end{pmatrix}. \quad (2.31)$$

Hence

$$\text{Tr}(D^2f) = B \text{ and } \det(D^2f) = -A^2 - AB. \quad (2.32)$$

Suppose that there exists a sequence of stable critical points $\{u_\epsilon\}$ such that $E_\epsilon(u_\epsilon) \leq C|\log \epsilon|$, and, up to extraction, n limiting vortices b_1, b_2, \dots, b_n . From Theorem 2.1, necessarily $K_{\mathcal{M}}(b_i) \leq 0$ for all i . In particular, none of vortices are at infinity since $K_{\mathcal{M}}$ is positive at the north pole of a surface of revolution with $\beta''(0) \neq 0$. Thus we can use (2.30) as the parametrization on \mathcal{M} .

However, when $K_{\mathcal{M}} \leq 0$, we have $\text{Tr}(D^2f) = B = -\frac{\alpha^2(S(\phi))}{r^2}K_{\mathcal{M}} \geq 0$, and $\det(D^2f) \leq 0$. If there exists a b_i such that $\det(D^2f)(b_i) < 0$, then $D^2f(b_i)$ must have a negative eigenvalue. On the other hand, assume that $\det(D^2f)(b_i) = 0$ for all i . We observe from (2.32) that $\det(D^2f) = 0$ if and only if $A = 0$ i.e. $\alpha' = -1$ which implies that $B = 0$ for the principle curvature in $\hat{\theta}$ direction is 0. Hence in this case, $D^2f(b_i) = 0$ for all i , and the second variation of W only involves second derivatives of the log term given by

$$-\pi \frac{d^2}{dt^2} \Big|_{t=0} \sum_{i \neq j} d_i d_j \log |b_i - b_j - t(V_i - V_j)| = \pi \sum_{i \neq j} d_i d_j \quad (2.33)$$

if we choose $V_i = b_i$. Then $\sum_{i=1}^n d_i = 0$ implies that

$$\sum_{i \neq j} d_i d_j = -\frac{1}{2} \sum_{i=1}^n d_i^2 < 0.$$

We have proved :

Theorem 2.3. *Let \mathcal{M} be a surface of revolution satisfying (2.26), and $\{u_\epsilon\}$ be a family of nonconstant solutions to (2.3) such that $E_\epsilon(u_\epsilon) \leq C|\log \epsilon|$. Then for ϵ small enough, u_ϵ is unstable.*

Remark 2.3. From Serfaty's result ([10]) on bounded simply connected domains in \mathbb{R}^2 and the example of surfaces of revolution, we conjecture that Theorem 2.3 holds for any simply connected compact surface, regardless of curvature conditions.

3 Vortex Annihilation

In this section we look for conditions on a manifold that will imply the ultimate annihilation of vortices under the Ginzburg-Landau heat flow. To this end, let

(\mathcal{M}, g) be a smooth 2-manifold with boundary and consider the initial-boundary value problem

$$\begin{cases} u_t - \Delta_{\mathcal{M}} u = (1 - |u|^2)u & \text{in } \mathcal{M} \times \mathbb{R}_+ \\ u = e & \text{on } \partial\mathcal{M} \times \mathbb{R}_+ \\ u = u_0 & \text{on } \mathcal{M} \times \{t = 0\} \end{cases} \quad (3.1)$$

where for convenience we will associate \mathbb{C} with \mathbb{R}^2 and consider $u : \mathcal{M} \times \mathbb{R}_+ \rightarrow \mathbb{R}^2$. Here e is a constant unit vector and the initial data u_0 is allowed to have any number of vortices as long as their total degree satisfies $\sum d_i = 0$.

Existence and regularity are standard for this problem:

Proposition 3.1. *If $u_0 \in W^{k,p}(\mathcal{M})$ with $k > 2 + \frac{2}{p}$ for some $1 \leq p < \infty$ and $u_0 = e$ on $\partial\mathcal{M}$, then (3.1) has a solution that exists for all time that is uniformly bounded. Furthermore, for each $T > 0$,*

$$\|u\|_{C([0,T], W^{k,p}(\mathcal{M}))} \leq C(\|u\|_{L^\infty}) \quad (3.2)$$

$$\|u\|_{C^1([0,T], W^{k-2,p}(\mathcal{M}))} \leq C(\|u\|_{L^\infty}) \quad (3.3)$$

Proof. This follows from Proposition 4.2 and 4.3 in [13]. \square

From the gradient flow structure of (3.1) we also easily establish:

Proposition 3.2. *For each $T > 0$,*

$$\begin{aligned} \int_0^T \int_{\mathcal{M}} |u_t|^2 dv_g + \int_{\mathcal{M}} \left[\frac{\|\nabla_g u\|_g^2}{2} + V(u) \right](\cdot, T) dv_g \\ = \int_{\mathcal{M}} \left[\frac{\|\nabla_g u_0\|_g^2}{2} + V(u_0) \right] dv_g \end{aligned} \quad (3.4)$$

where $V(u) = \frac{1}{4}(1 - |u|^2)^2$.

Proof. Taking inner product of (3.1)-1 with u_t and integrating over \mathcal{M} for a fixed t , we have

$$\begin{aligned} \int_{\mathcal{M}} |u_t|^2 dv_g &= \int_{\mathcal{M}} u_t \cdot [\Delta_{\mathcal{M}} u + (1 - |u|^2)u] dv_g \\ &= - \int_{\mathcal{M}} \langle \nabla_g u_t, \nabla_g u \rangle_g + V(u)_t dv_g \\ &= - \int_{\mathcal{M}} \left[\frac{\|\nabla_g u\|_g^2}{2} + V(u) \right]_t dv_g \end{aligned}$$

Integrating from 0 to T , we get the desired equality. \square

Lemma 3.1. *Suppose $p > 2$ and u_0 satisfies the assumption of Proposition 3.1. Then for any sequence $\{t_n\}$ with $t_n \rightarrow \infty$ as $n \rightarrow \infty$, there exists a subsequence $\{t_{n_j}\}$ and a function \bar{u} such that*

$$u(\mathbf{x}, t_{n_j}) \rightarrow \bar{u}(\mathbf{x}) \quad \text{in } C^2(\bar{\mathcal{M}}),$$

and

$$\begin{cases} -\Delta_{\mathcal{M}} \bar{u} = (1 - |\bar{u}|^2)\bar{u} & \text{in } \mathcal{M} \\ \bar{u} = e & \text{on } \partial\mathcal{M} \end{cases} \quad (3.5)$$

Proof. From (3.2) of Proposition 3.1, the sequence $\{u(\cdot, t_n)\}$ is uniformly bounded in $W^{k,p}$. So by the Sobolev embedding theorem, there is a subsequence $\{t_{n_j}\}$ and a C^2 function $\bar{u}(\mathbf{x})$ such that

$$u(\mathbf{x}, t_{n_j}) \rightarrow \bar{u}(\mathbf{x}) \quad \text{in } C^2(\bar{\mathcal{M}}).$$

To prove $\bar{u}(\mathbf{x})$ is a solution of (3.5), first we show that $\lim_{t \rightarrow \infty} \|u_t\|_{L^\infty(\mathcal{M})} = 0$. Assume by way of contradiction that there is a sequence $\{(\mathbf{x}_n, t_n)\}$ with $t_n \rightarrow \infty$ such that $|u_t(\mathbf{x}_n, t_n)| > \epsilon > 0$. Since (3.3) of Proposition 3.1 implies that u_t is uniformly continuous, there exists a $\delta > 0$ so that for all n , we have

$$|u_t(\mathbf{x}, t)| > \frac{\epsilon}{2} \quad \text{for } (\mathbf{x}, t) \in B_\delta(\mathbf{x}_n) \times (t_n - \delta, t_n + \delta),$$

where $B_\delta(\mathbf{x}_n)$ is the geodesic ball in \mathcal{M} centered at \mathbf{x}_n with radius δ . But then $\int_0^\infty \int_{\mathcal{M}} |u_t|^2 dv_g = \infty$ which contradicts Proposition 3.2. Now, taking the limit as $j \rightarrow \infty$ in (3.1)-1 at time t_{n_j} , we get (3.5). \square

Lemma 3.2. *Suppose v solves (3.5) such that $v(\mathbf{x}) \neq 0$ for all \mathbf{x} in \mathcal{M} . Then v is a constant.*

Proof. Write $e = e^{i\alpha_0}$ for some $\alpha_0 \in [0, 2\pi)$. Then from the assumption $v(\mathbf{x}) \neq 0$, we may write the function $\tilde{v} = ve^{-i\alpha_0}$ in the form

$$\tilde{v} = \rho e^{i\alpha}$$

for some smooth functions $\rho : \mathcal{M} \rightarrow \mathbb{R}_+$, and $0 \leq \alpha < 2\pi$. Plugging this form into (3.5), we have

$$\begin{cases} \Delta_{\mathcal{M}} \rho - \rho |\nabla_g \alpha|_g^2 = \rho^3 - \rho & \text{in } \mathcal{M} \\ \rho \Delta_{\mathcal{M}} \alpha + 2\langle \nabla_g \rho, \nabla_g \alpha \rangle_g = 0 & \text{in } \mathcal{M} \\ \rho = 1 & \text{on } \partial\mathcal{M} \\ \alpha = 0 & \text{on } \partial\mathcal{M} \end{cases} \quad (3.6)$$

Multiplying (3.6)-2 by $\alpha\rho$ and integrating, we get

$$\begin{aligned} 0 &= \int_{\mathcal{M}} \alpha [\rho^2 \Delta_{\mathcal{M}} \alpha + 2\rho \langle \nabla \rho, \nabla \alpha \rangle_g] dv_g \\ &= \int_{\mathcal{M}} \alpha \operatorname{div}(\rho^2 \nabla \alpha) dv_g \\ &= - \int_{\mathcal{M}} \rho^2 |\nabla \alpha|_g^2 dv_g \end{aligned}$$

Thus $\nabla \alpha = 0$ on \mathcal{M} , and so $\alpha \equiv 0$.

Then from (3.6)-1, since $\rho = |v| \geq 0$, we conclude by the maximum principle that $\rho \equiv 1$ on \mathcal{M} . This proves the lemma. \square

In the last section, we deduced that the linear instability of nonconstant critical points of the Ginzburg-Landau energy for a surface of revolution is independent of any curvature assumptions. Now we will derive a result that is similar in spirit for the parabolic problem (3.1) posed on a surface of revolution. Consider a surface \mathcal{M} with boundary defined parametrically as in the last section:

$$\mathbf{x}(s, \theta) = (\alpha(s) \cos(\theta), \alpha(s) \sin(\theta), \beta(s)), \quad 0 \leq s \leq l, \quad 0 \leq \theta \leq 2\pi$$

with $\alpha(0) = \beta(0) = \beta'(0) = 0$, and $\alpha(s) > 0$ for $s \neq 0$. Note that $\partial\mathcal{M} = \{\mathbf{x}(l, \theta) : 0 \leq \theta \leq 2\pi\}$. We recall that the induced metric is

$$g = ds^2 + \alpha^2(s) d\theta^2.$$

Now we present a crucial lemma which one can view as a kind of parabolic Pohozaev identity for heat flow on a manifold, cf. [2], Lemma 4.1.

Lemma 3.3. *Let $H(s) = \int_0^s \alpha(\tilde{s}) d\tilde{s}$, and let $\tilde{H} : \mathcal{M} \rightarrow \mathbb{R}$ be defined for any $p = \mathbf{x}(s, \theta) \in \mathcal{M}$ by the relation $\tilde{H}(p) = H(s)$. Then for each $T > 0$,*

$$\begin{aligned} \int_0^T \int_{\mathcal{M}} \tilde{H} |u_t|^2 + (\Delta_{\mathcal{M}} \tilde{H}) V(u) dv_g dt + \int_{\mathcal{M}} \tilde{H} \left[\frac{\|\nabla_g u\|_g^2}{2} + V(u) \right] (\cdot, T) dv_g \\ \leq \int_{\mathcal{M}} \tilde{H} \left[\frac{\|\nabla_g u_0\|_g^2}{2} + V(u_0) \right] dv_g. \end{aligned} \quad (3.7)$$

Proof. First, taking the inner product of (3.1)-1 with $\tilde{H} u_t$ and integrating over \mathcal{M} for a fixed t , we have

$$\begin{aligned} \int_{\mathcal{M}} \tilde{H} |u_t|^2 dv_g &= - \int_{\mathcal{M}} \langle \nabla_g u, \nabla_g (\tilde{H} u_t) \rangle_g dv_g - \int_{\mathcal{M}} \tilde{H} V(u)_t dv_g + \int_{\partial\mathcal{M}} \frac{\partial u}{\partial \mathbf{n}} \cdot \tilde{H} u_t \\ &= - \int_{\mathcal{M}} \tilde{H} \left[\frac{\|\nabla_g u\|_g^2}{2} + V(u) \right] dv_g - \int_{\mathcal{M}} u_t \cdot \langle \nabla_g u, \nabla_g \tilde{H} \rangle_g dv_g, \end{aligned} \quad (3.8)$$

where $\langle \nabla_g u, \nabla_g v \rangle_g \equiv \sum_{i=1}^2 \langle \nabla_g u_i, \nabla_g v_i \rangle_g$ for functions $u = (u_1, u_2), v = (v_1, v_2)$ in \mathbb{R}^2 . Here we have used the boundary condition $u = e$ on $\partial\mathcal{M} \times \mathbb{R}_+$, to chop the boundary integral.

Next, using (3.1)-1 and integrating by parts, we obtain

$$\begin{aligned} - \int_{\mathcal{M}} u_t \cdot \langle \nabla_g u, \nabla_g \tilde{H} \rangle_g dv_g \\ &= - \int_{\mathcal{M}} \Delta_{\mathcal{M}} u \cdot \langle \nabla_g u, \nabla_g \tilde{H} \rangle_g dv_g - \int_{\mathcal{M}} (1 - |u|^2) u \cdot \langle \nabla_g u, \nabla_g \tilde{H} \rangle_g dv_g \\ &= - \int_{\mathcal{M}} \Delta_{\mathcal{M}} u \cdot \langle \nabla_g u, \nabla_g \tilde{H} \rangle_g dv_g + \int_{\mathcal{M}} \langle \nabla_g V(u), \nabla_g \tilde{H} \rangle_g dv_g \\ &= - \int_{\mathcal{M}} \Delta_{\mathcal{M}} u \cdot \langle \nabla_g u, \nabla_g \tilde{H} \rangle_g dv_g - \int_{\mathcal{M}} (\Delta_{\mathcal{M}} \tilde{H}) V(u) dv_g \end{aligned} \quad (3.9)$$

Integrating by parts twice in the first term on the right hand side yields

$$\begin{aligned}
& - \int_{\mathcal{M}} \triangle_{\mathcal{M}} u \cdot \langle \nabla_g u, \nabla_g \tilde{H} \rangle_g dv_g \\
& = \int_{\mathcal{M}} \frac{1}{2} \langle \nabla_g ||\nabla u||_g^2, \nabla_g \tilde{H} \rangle_g + \alpha' ||\nabla_g u||_g^2 dv_g - \int_{\partial\mathcal{M}} \frac{\partial u}{\partial \mathbf{n}} \cdot \langle \nabla_g u, \nabla_g \tilde{H} \rangle_g dS \\
& = \frac{1}{2} \int_{\partial\mathcal{M}} ||\nabla_g u||_g^2 \langle \nabla_g \tilde{H}, \mathbf{n} \rangle_g dS - \int_{\partial\mathcal{M}} \frac{\partial u}{\partial \mathbf{n}} \cdot \langle \nabla_g u, \nabla_g \tilde{H} \rangle_g dS \tag{3.10}
\end{aligned}$$

Note that $\nabla_g \tilde{H} = \alpha \mathbf{n}$ on $\partial\mathcal{M}$, (3.10) can be rewritten as

$$\begin{aligned}
& \frac{1}{2} \int_{\partial\mathcal{M}} ||\nabla_g u||_g^2 \langle \nabla_g \tilde{H}, \mathbf{n} \rangle_g dS - \int_{\partial\mathcal{M}} \frac{\partial u}{\partial \mathbf{n}} \cdot \langle \nabla_g u, \nabla_g \tilde{H} \rangle_g dS \\
& = \frac{1}{2} \int_{\partial\mathcal{M}} \alpha \left[\left| \frac{\partial u}{\partial \tau} \right|_g^2 - \left| \frac{\partial u}{\partial \mathbf{n}} \right|_g^2 \right] dS \\
& = -\frac{1}{2} \int_{\partial\mathcal{M}} \alpha \left| \frac{\partial u}{\partial \mathbf{n}} \right|_g^2 dS \tag{3.11}
\end{aligned}$$

Combining (3.8)-(3.11), since $\alpha \geq 0$, we have

$$\int_{\mathcal{M}} \tilde{H} |u_t|^2 + (\triangle_{\mathcal{M}} \tilde{H}) V(u) dv_g + \int_{\mathcal{M}} \tilde{H} \left[\frac{||\nabla_g u||_g^2}{2} + V(u) \right] dv_g \leq 0$$

Integrating from 0 to T gives the desired inequality. \square

Theorem 3.1. Assume $p > 2$ and u_0 satisfies the assumption of Proposition 3.1. If $\alpha'(s) \geq c > 0$ for $0 \leq s \leq l$, then $u(x, t) \rightarrow e$ uniformly as $t \rightarrow \infty$. In particular, u has no vortices after some finite time T .

Proof. Since $\triangle_{\mathcal{M}} \tilde{H} = 2\alpha'(s) \geq c > 0$, from Lemma 3.3, we have

$$\int_0^\infty \int_{\mathcal{M}} V(u) dv_g dt < \infty.$$

Then arguing as in Lemma 3.1, we obtain

$$\lim_{t \rightarrow \infty} |u(x, t)| = 1 \text{ uniformly for } x \in \mathcal{M}. \tag{3.12}$$

Now, (3.12) and Lemma 3.2 implies that $||u(\cdot, t) - e||_{L^\infty} \rightarrow 0$ \square

Remark 3.1. From Proposition 3.1 there exists a $\delta > 0$ independent on t such that $|u| > \frac{1}{2}$ in $\{\mathbf{x}(s, \theta) : l - \delta < s \leq l, 0 \leq \theta \leq 2\pi\}$. Thus the condition of α can be replaced by $\alpha'(s) \geq c > 0$ for $0 \leq s \leq l - \delta$ in the theorem.

Remark 3.2. The argument in the theorem does not involve the second derivative of α . This indicates that the curvature does not affect the large-time behavior of the solution for this type of surface.

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